CHAPTER 5
Hydrostatic Transmissions (Hydrostatic Drives)

"Hydrostatic transmission" is the name given to a hydraulic system consisting of a positive displacement pump and a positive displacement motor. A wider definition of hydrostatic transmissions includes hydraulic systems consisting of several pumps and/or motors and even systems in which the output devices are actuators.

The purpose of hydrostatic transmission is to convert mechanical power into hydraulic power then to transmit and convert it back into mechanical output power in a form which matches speed and torque demands of a driven mechanism or machine.

In the previous chapter we discussed two main methods of speed control, namely:

- *Resistive (valve speed control)* - where the control of flow, thus the speed of output units, is achieved by flow control valves. The pump and motor are fixed (constant) displacement units.
- *Volumetric (pump/motor control)* - where speed is controlled by varying the displacement parameters of a pump, motor or both.

Valve speed control of hydrostatic transmissions, which has associated significant power losses, has limited application in the control of rotary, especially high power, hydrostatic transmissions. Its main application is in drives employing actuators as output units, in which continuous control of velocity is usually not required. As valve speed control was discussed in some detail in Section 4.2, we will discuss only volumetric control (pump/motor control) of rotary hydrostatic transmissions.

In hydrostatic transmissions control of the direction of rotation of a motor its speed of rotation is accomplished by adjusting displacement of pumps, motors or both. We may also control the direction of rotation of a motor, and to some extent its speed, by a directional control valve interposed between a pump and a motor.

Hydrostatic transmissions may operate in either *open* or *closed* circuits. An *open* circuit, shown in fig. 129, consists of a variable displacement pump **1**, a variable displacement motor **2** and a directional control valve **3** which controls the direction of fluid flow and thus direction of rotation of the motor. The speed of the motor is controlled either by changing displacements of rotary units. The return flow from the motor is directed by a directional control valve to the reservoir.

A *closed* circuit consists of a variable displacement pump **1** and a variable or fixed displacement motor **2,** the return flow from the motor is supplied to the inlet port of the pump. In this system change of the direction of rotation of motor **2** is obtained by using an *over the centre* pump **1** capable of reversing the flow direction, fig. 130. Control of speed of the motor is accomplished by changing displacement of the

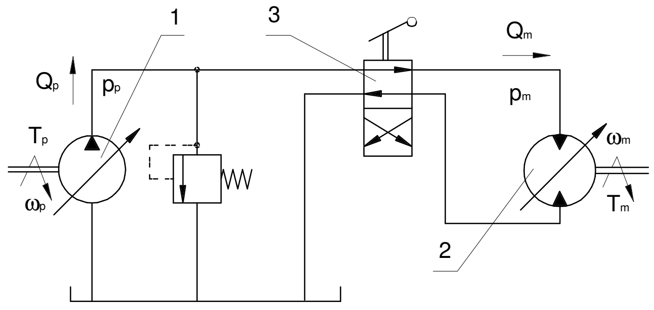

Fig. 129. Open circuit hydrostatic transmission

pump or the motor or both. "Cross-line" relief valves **6** and **7** protect the system against excessive pressure. The volumetric losses in the system (internal and external leakages) are replenished by an additional charge pump **3**, which is protected by a low pressure relief valve **8**, or by an accumulator. The replenishing flow is directed, via check valves **4** or **5**, to this branch of the system which, depending on the direction of flow, is operating at low pressure.

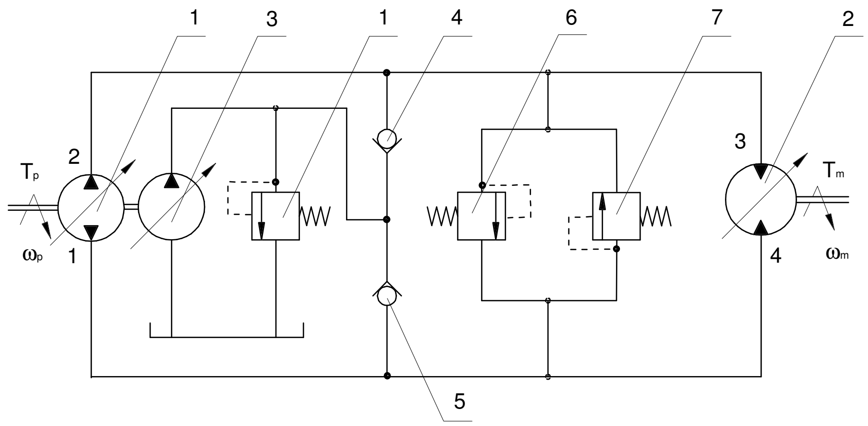

Fig. 130. Closed circuit hydrostatic transmission

Transformation of power transmitted by a hydrostatic transmission is described by two relations; dynamic ratio or kinematic ratio. Dynamic ratio i_d defines how the transmission changes torque:

$$i_d = \frac{T_2}{T_1} \tag{5.1}$$

where:

T_1, T_2 - are correspondingly input and output torques.

Kinematic ratio i_k defines how the transmission transforms input to output speed:

$$i_k = \frac{n_2}{n_1} = \frac{\omega_2}{\omega_1} \tag{5.2}$$

where:

ω_1, n_1 - angular or rotation speed of input shaft

ω_2, n_2 - angular or rotation speed of input shaft

As the efficiency of transmission is a ratio of output power P_2 on the motor shaft to input power P_1 delivered to the pump shaft, then:

$$\eta = \frac{P_2}{P_1} = \frac{T_2\omega_2}{T_1\omega_1} = i_d i_k \tag{5.3}$$

thus, the efficiency of the transmission can be expressed as a product of dynamic and kinematic ratios.

For a transmission in which the above ratios are variable, it is important to define the transmission range, i.e. a permissible range of dynamic ratio i_d and a range of kinematic ratio i_k. In practice, permissible ranges of i_d and i_k ratios are defined using criteria that over full range of these transmission ratios the transmission will operate with efficiency above $\eta = 0.8$.

5.1 Characteristics of an ideal hydrostatic transmission

Analysis of operating parameters of a hydrostatic transmission, when the transmission works under varying load and various control strategies are employed, is difficult due to a large number of variable parameters and their interaction. However, if we assume that the transmission operates without losses ($\eta = 1$), then using basic relations which describe the operation of a pump and a motor we may investigate with relative ease the effect of various parameters on the performance of the transmission.

If we assume that a hydraulic system operates without losses, basic parameters of the system can be calculated using equations given in chapter 2. Pressure differential Δp_{tm} across the motor, under assumption of no pump/motor or line losses and steady-state operating conditions, is the result of load torque T_{load}:

$$\Delta p_{tm} = \frac{2\pi T_{load}}{\varepsilon_m q_m} \tag{5.4}$$

or

$$\Delta p_{tm} = \frac{T_{load}}{\varepsilon_m V_{\phi m}} \tag{5.5}$$

where:

$q_m, V_{\phi m}$ - stroke and unit displacements of the motor

ε_m - displacement parameter

In a steady-state, torque T_{tm} is equal to load torque T_{load} and as we have ignored line losses then $\Delta p_{tp} = \Delta p_{tm}$ and thus pump torque T_{tp} is equal to:

$$T_{tp} = \frac{\varepsilon_p q_p}{2\pi} \Delta p_{tp} \tag{5.6}$$

or

$$T_{tp} = \varepsilon_p V_{\phi p} \Delta p_{tp} \tag{5.7}$$

and using eq. (5.4) (or eq. (5.5)) and $\Delta p_{tp} = \Delta p_{tm}$, $T_{tm} = T_{load}$ the relation between pump and motor torques is expressed by:

$$T_{tp} = \frac{\varepsilon_p V_{\phi p}}{\varepsilon_m V_{\phi m}} T_{tm} \tag{5.8}$$

Pump delivery flow Q_{tp}, under assumption of no leakages, is described by equation:

$$Q_{tp} = \varepsilon_p q_p n_p \tag{5.9}$$

or

$$Q_{tp} = \varepsilon_p V_{\phi p} \omega_p \tag{5.10}$$

this flow will be delivered to the motor, thus $Q_{tm} = Q_{tp}$. As the theoretical flow demand of the motor is described by equation:

$$Q_{tm} = \varepsilon_m q_m n_m = \varepsilon_m V_{\phi m} \omega_m \tag{5.11}$$

then using eq. (5.9) (or eq. (5.10)) the rotation speed of the motor is defined by the relation:

$$n_m = \frac{\varepsilon_p q_p}{\varepsilon_m q_m} n_p \tag{5.12}$$

or

$$\omega_m = \frac{\varepsilon_p V_{\varphi p}}{\varepsilon_m V_{\varphi m}} \omega_p \tag{5.13}$$

We may differentiate between the following types of controls for hydrostatic transmissions:

- Constant torque, variable power
- Constant power, variable torque
- Variable power, variable torque

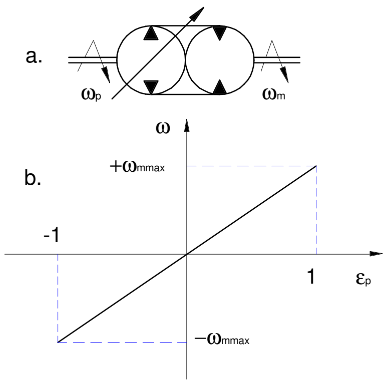

Fig. 131. Hydrostatic transmission with a fixed displacement motor: a. graphic symbol, b. speed characteristic of the motor

5.1.1 Constant torque, variable power (ε_p varying, ε_m = constant)

In this type of control of a hydrostatic transmission, a variable displacement pump supplies fluid to a fixed displacement motor. The speed of the motor is controlled by changing the displacement of the pump. In most cases displacement parameter ε_p varies between $-1 \leq \varepsilon_p \leq 1$ but motor displacement parameter $\varepsilon_m = 1$. The angular speed of the motor when all parameters, with the exception of ε_m, are constant is defined as:

$$\omega_m = K\varepsilon_p \quad \text{where } K = const. \tag{5.14}$$

and is thus a linear function of ε_p, fig. 131b.

5.1.2 Constant power, variable torque (ε_p constant, ε_m variable)

In this type of transmission the pump has a fixed displacement, i.e. $\varepsilon_p = 1$ but displacement of a hydraulic motor varies in range $-1 \leq \varepsilon_m \leq 1$ and thus the angular speed of the motor, assuming that all parameters with the exception of ε_m are constant, can be expressed as:

$$\omega_m = \frac{K}{\varepsilon_m} \text{ where } K = const. \tag{5.15}$$

thus the speed of the motor is a hyperbolic function of ε_m, as shown in fig. 132b.

5.1.3 Variable power, variable torque (ε_p and ε_m variable)

Transmissions in which displacements of both pump and motor may take any value, fig. 133, can be controlled sequentially or simultaneously. In simultaneously controlled transmissions both ε_p and ε_m are varied according to some program, usually

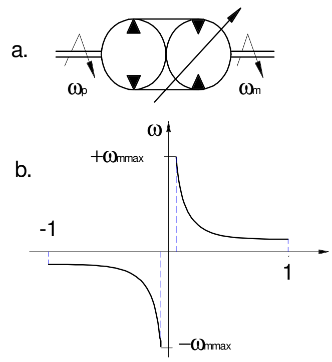

Fig. 132. Hydrostatic transmission with a fixed displacement pump: a. graphic symbol, b. speed characteristic of the motor

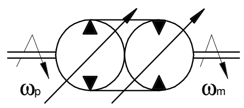

Fig. 133. Hydrostatic transmission with a variable displacement pump and a variable displacement motor - graphic symbol

pump displacement parameter ε_p is controlled first and then motor displacement parameter ε_m. The sequential control of a transmission is shown in fig. 134, and to illustrate its operation we will discuss three control cases:

- The motor is initially set at its maximum displacement ($\varepsilon_m = 1$) and the pump at its minimum displacement ($\varepsilon_p = 0$). The straight line **OA** represents change of motor speed ω_m with an increase of parameter ε_p from zero to $\varepsilon_p = 1$. When the pump is at its maximum displacement (point **A**), further control of motor speed is achieved by reducing motor displacement parameter ε_m. Curve **AG** shows the change of motor speed with variation of displacement parameter ε_m.
- The motor is set at $\varepsilon_m = 0.5$ (i.e. at half displacement) and the pump is set at zero displacement, $\varepsilon_p = 0$. The motor speed as a function of pump displacement, when it increases from $\varepsilon_p = 0$ to $\varepsilon_p = 1$, is represented by straight line **OE**. When the pump reaches it maximum displacement $\varepsilon_p = 1$, i.e. its maximum flow, further variation of motor speed ω_m is obtained by decreasing the motor displacement, by curve **FG**.
- When the motor is set at maximum displacement ($\varepsilon_m = 1$) and the pump displacement increases from $\varepsilon_p = 0$ to $\varepsilon_p = 0.5$, then straight line **OB** represents the change of motor speed. In this case reduction of motor displacement ε_m and the corresponding increase of motor speed is represented by curve **CD**.

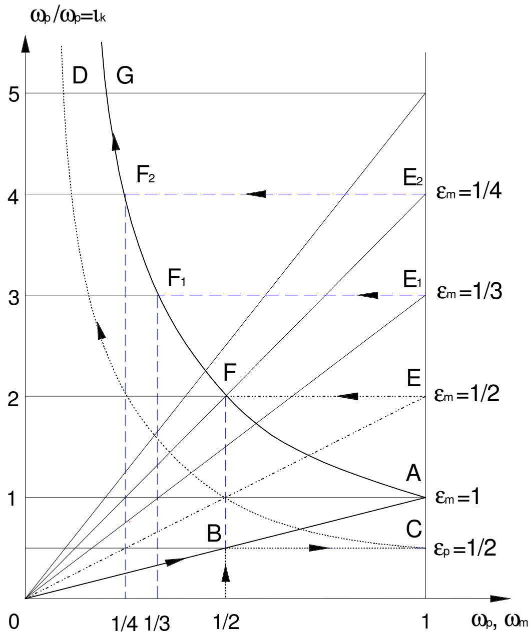

Fig. 134. Sequential control of an ideal hydrostatic drive

If we assume that a hydrostatic transmission should operate in the range for which its efficiency is above $\eta = 0.8$ the sequential control of the transmission is preferable as it allows obtaining the widest control range, kinematic ratio i_k and dynamic ratio i_d is $6 \div 7$. When only the pump displacement is controlled, the maximum range of dynamic ratio is $i_d = 4.5$ and the range of kinematic ratio is $i_d = 4.0$.

To derive operational parameters of an ideal transmission (no losses) we use equations for pump input power P_p and motor output torque T_m:

$$P_p = \varepsilon_p V_{\phi p} \omega_p \Delta p \tag{5.16}$$

$$T_m = \varepsilon_m V_{\phi m} \Delta p \tag{5.17}$$

Equation (5.17) shows that for a given motor, output torque depends on displacement parameter ε_m and pressure differential across the motor Δp. Thus when the motor displacement is set, i.e. $\varepsilon_m = const.$, the motor output torque is proportional to pressure differential Δp across the motor, fig. 135a. The straight lines for various values of $\varepsilon_m = const.$ pass through the origin. Plotted in logarithmic coordinates these lines become parallel to each other, fig. 135b.

When we reduce parameter ε_m but maintain a constant torque ($T_m = const.$) then pressure differential Δp across the motor will increase, arrow **1** in fig. 135b. On the other hand, we may increase the output torque at a set pressure differential ($\Delta p = const.$) by increasing displacement parameter ε_m, arrow **2** in fig 135b.

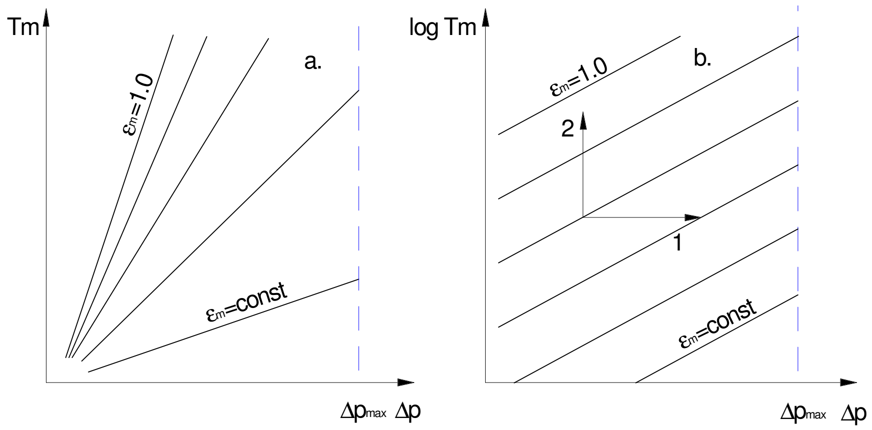

Fig. 135. Characteristics of a variable displacement motor $T_m = f(\Delta p, \varepsilon_m)$. a. Linear co-ordinates, b. Logarithmic co-ordinates.

If we consider eq. (5.16) we can see that pump input power P_p is proportional to pump parameter ε_p and pressure difference Δp. Thus plots of $P_p = const.$ in $\Delta p, \varepsilon_p$ coordinates are hyperbolas fig. 136a., and in logarithmic coordinates these curves are straight lines, fig. 136b.

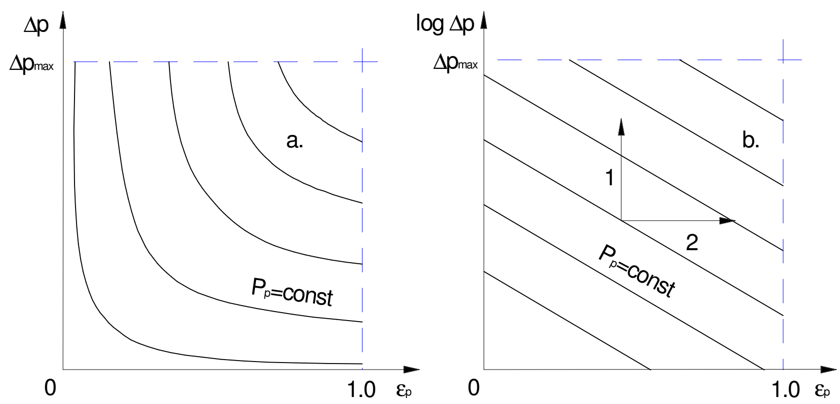

Fig. 136. Characteristics of a variable displacement pump $P_p = f(\Delta p, \varepsilon_p)$. a. linear co-ordinates, b. logarithmic co-ordinates.

However, when we maintain parameter $\varepsilon_p = const.$, then pump input power P_p will increase with an increase in pressure differential Δp, arrow **1** in fig. 136b. On the other hand, if pressure differential Δp is maintained constant then the required pump input power P_p will increase with an increase of parameter ε_p , i.e. with an increase of the pump flow Q_p, as shown by arrow **2** in fig. 136b.

To find pressure differential Δp for an ideal hydrostatic transmission, we assume that $\Delta p = \Delta p_{12} = \Delta p_{34}$ and use the equation for motor output torque:

$$\Delta p = \frac{T_m \omega_m}{\varepsilon_p V_{\phi p} \omega_p} \tag{5.18}$$

From the above equation we can see that for a transmission operating at a set pressure differential ($\Delta p = const.$) and a set value of the motor output torque ($T_m = const.$), the pressure differential Δp is proportional to the angular speed of motor ω_m. Thus at constant pump delivery, in ω_m, Δp coordinates, plots of the output torque are straight lines crossing the origin, fig. 137a. For other displacement settings of the pump, straight lines $T_m = const.$ are rotated in relation to the previous lines by an angle corresponding to the change of parameter ε_p.

At a constant setting of pump displacement parameter ε_p and a constant value of torque T_m, the increase of the angular speed of motor ω_m, obtained by reducing motor displacement ε_m, is accompanied by the increased pressure difference Δp across the motor, arrow **1** - fig. 137b. On the other hand, increasing the angular speed ω_m, at constant values of Δp and ε_p, causes a decrease of the torque which can be developed by the motor, arrow **2** - fig. 137b.

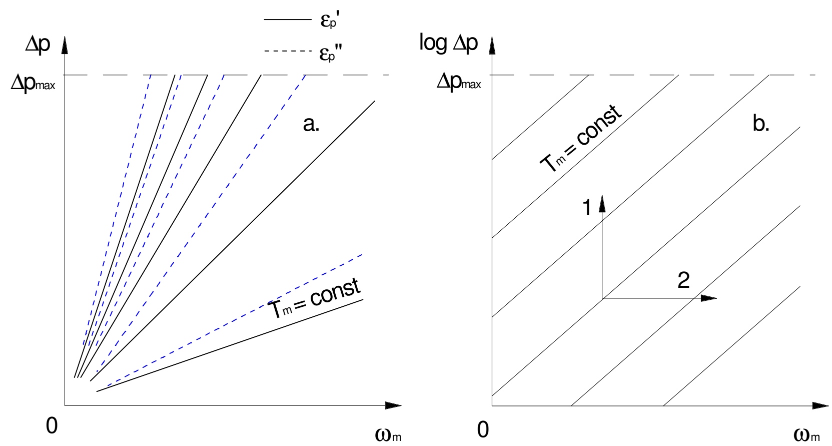

Fig. 137. Characteristics of hydrostatic transmission $\Delta p = f(\omega_m, T_m, \varepsilon_m)$. Lines T_m = const (at Δp = const). a. linear co-ordinates, b. logarithmic co-ordinates.

It must be stated, however, that the operational parameters of a real transmission differ from the parameters derived for an ideal transmission. This is specially the case when the system is operating with the pump set at a small displacement, large motor speed or when the system operates at either very high or very low pressures.

5.2 Volumetric and hydro-mechanical losses

5.2.1 Volumetric losses

Hydrostatic transmissions suffer volumetric losses Q_v which occur both in pump and motor units. Volumetric losses (discussed in Chapter 2) are for a given speed mainly a function of load, i.e. pressure Δp, in a system:

$$Q_v = K_v \Delta p \tag{5.19}$$

and are expressed in terms of the volumetric efficiencies of pump η_p and motor η_m, which were defined by equations:

$$\eta_{vp} = \frac{Q_p}{Q_{tp}} = \frac{Q_p}{\varepsilon_p \omega_p V_{\phi p}} \tag{5.20}$$

and

$$\eta_{vm} = \frac{Q_{tm}}{Q_m} = \frac{\varepsilon_m \omega_m V_{\phi m}}{Q_m} \tag{5.21}$$

The above equations show that displacement settings ε_p and ε_m have influence on efficiencies of both the pump and the motor. In addition, some volumetric losses also occur in control mechanisms of the hydrostatic units. These losses are taken into consideration when calculating transmission efficiency only if the control elements are supplied from the main circuit of the transmission, e.g. displacement or pressure controls, by defining efficiency η_{vz} of the control system. The volumetric efficiency of the transmission lines for correctly manufactured systems is $\eta_{vl} = 1.0$ i.e. there is no line leakage. Comparing actual demand of motor Q_m with pump delivery Q_p we obtain expression for the angular speed of the motor:

$$\omega_m = \frac{\varepsilon_p V_{\phi p}}{\varepsilon_m V_{\phi m}} \omega_p \eta_{vp} \eta_{vm} \eta_{vz} \tag{5.22}$$

We can see that motor speed is lower than the speed of an ideal transmission operating without volumetric losses. Kinematic ratio i_k for a transmission with volumetric losses is equal to:

$$i_k = \frac{\omega_m}{\omega_p} = \frac{\varepsilon_p V_{\phi p} \eta_{vm} \eta_{vz} \eta_{vp}}{\varepsilon_m V_{\phi m}} \tag{5.23}$$

$$i_k = \frac{\eta_{vm} \eta_{vz} \eta_{vp}}{i} \tag{5.24}$$

where i is transmission ratio equal to:

$$i = \frac{\varepsilon_m V_{\phi m}}{\varepsilon_p V_{\phi p}} \tag{5.25}$$

Thus the kinematic ratio of the transmission will diminish when volumetric losses of the hydrostatic units increase.

5.2.2 Hydro-mechanical losses

Hydro-mechanical losses are defined by hydro-mechanical efficiencies of the individual elements of the transmission. The efficiency of pump η_{hmp} and the efficiency of motor η_{hmm} were defined as:

$$\eta_{hmp} = \frac{T_{tp}}{T_p} \qquad \text{and} \qquad \eta_{hmm} = \frac{T_m}{T_{tm}} \tag{5.26}$$

Additional hydro-mechanical losses are due to pressure losses Δp_c in the control elements of the transmission and pressure losses Δp_l due to friction in the delivery and return lines of the transmission.

Hydraulic efficiency η_{hl} of the hydraulic line is defined as a ratio of pressure difference $(p_3 - p_4)$ at the motor ports 3 and 4 to pressure difference $(p_2 - p_1)$ at the pump ports 1 and 2, fig. 130:

$$\eta_{hl} = \frac{p_3 - p_4}{p_2 - p_1} = \frac{\Delta p_{34}}{\Delta p_{21}} \tag{5.27}$$

where pressure p_3 includes pressure loss Δp_l in the hydraulic line and losses Δp_c in the control elements:

$$p_3 = p_2 - \Delta p_l - \Delta p_c \tag{5.28}$$

Thus, the hydraulic efficiency of the line is:

$$\eta_{hl} = \frac{p_2 - \Delta p_l - \Delta p_c - p_4}{p_2 - p_1} \tag{5.29}$$

Pressure loss Δp_l is a function of the flow rate Q and fluid viscosity μ, thus $\Delta pl = f(Q, \mu)$.

Dynamic ratio i_d of the transmission can now be defined as:

$$i_d = \frac{T_m}{T_p} = \frac{T_{tm}\eta_{hmm}\eta_{hmp}}{T_{tp}} \tag{5.30}$$

substituting for T_m and T_p, we obtain:

$$i_d = \frac{\varepsilon_m V_{\phi m} \Delta p_{34}}{\varepsilon_p V_{\phi p} \Delta p_{21}} \eta_{hmp}\eta_{hmm} \tag{5.31}$$

or:

$$i_d = i\eta_{hmp}\eta_{hmm}\eta_{hl} \tag{5.32}$$

Thus the hydro-mechanical efficiencies of the pump and motor, control elements and hydraulic lines decrease the dynamic ratio of a transmission.

5.2.3 Effect of load

In general, steady-state analysis of hydraulic systems is carried out under the assumption of incompressibility of working fluid, which is reflected in an instant and exact reaction of a system to parametric changes. The real hydraulic fluid is, however, non-homogeneous - it contains free air and a mixture of liquid and air, thus it is necessary to consider the effects of fluid compressibility on system performance. This is especially important in the case of systems which must provide exact load positioning and fast response to control signals.

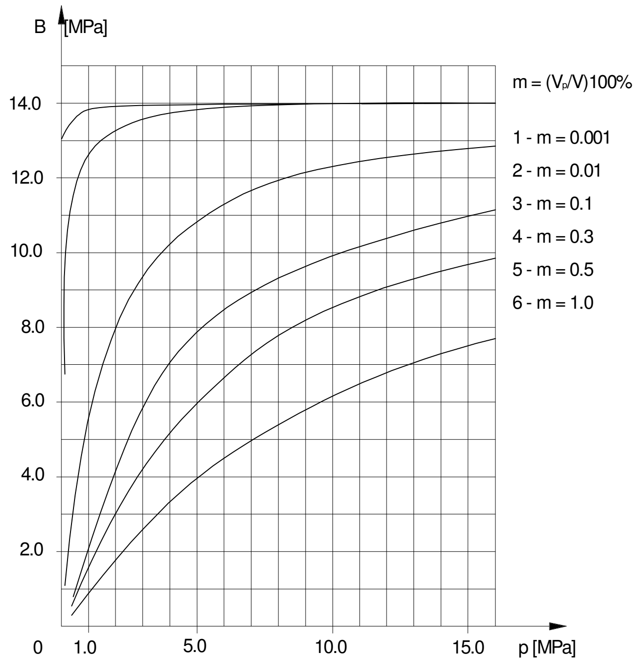

Fig. 138. Dependence of bulk modulus on amount of air and operating pressure. m - air content (%) , V_p - air volume, V -liquid volume.

Compressibility of fluid is characterized by its bulk modulus. Bulk modulus depends on many factors, e.g. the type of hydraulic fluid, temperature, pressure and degree of aeration. The effect of aeration on the value of bulk modulus can be quite drastic. Fig. 138 shows how bulk modulus varies with pressure and the content of air in a typical hydraulic fluid. Bulk modulus B of the fluid allows the determination of stiffness coefficient C of a hydraulic system.

The stiffness of a hydraulic system is expressed as a ratio of load change ΔF (or ΔT) to the corresponding position change Δx (or $\Delta\alpha$) of the loaded element:

$$C = \frac{\Delta F}{\Delta x} \tag{5.33}$$

or

$$C = \frac{\Delta T}{\Delta \alpha} \tag{5.34}$$

High hydraulic stiffness is very important in systems driving machine tools, robots etc. The stiffness of a system is a function of the changes in fluid volume in the system which are due to the variation of system pressure. The change in fluid volume due to a change in pressure is defined by the relation:

$$V = V_0\left(1 - \frac{p}{B_a}\right) \tag{5.35}$$

where:

V_0, V	-	fluid volume at atmospheric pressure and at pressure p
B_a	-	apparent bulk modulus of the fluid

Apparent bulk modulus B_a combines ideal bulk modulus B with the effects of aeration and compliance of hydraulic lines and system elements (e.g. actuators). In initial design calculations we may assume the value of apparent bulk modulus of a mineral oil to be $B_a = 1200$ MPa. The compressibility of fluid and the compliance of the system (mainly of transmission lines) change the effective flow rate in the system. Thus, the effective flow rate in a hydrostatic system can be expressed by equation:

$$\varepsilon_p \omega_p V_{\phi p} - K_{vp}\Delta p - \frac{V_0}{B_a}\frac{d\Delta p}{dt} = \varepsilon_m \omega_m V_{\phi m} + K_{vm}\Delta p = Q \tag{5.36}$$

where:

K_{vp}	-	coefficients of volumetric loss in pump
K_{vm}	-	coefficients of volumetric loss in motor
Δp	-	load pressure, $\Delta p = f(T_{load})$

This is a basic equation for a hydrostatic transmission, it includes the effects of the compressibility of fluid and the volumetric losses in the pump and motor. This equation is true, regardless of the type of transmission and the type of loading, as long as the volumetric losses in the transmission line can be ignored. To allow analysis of the effect of load on the speed of the motor the above equation can be written in the form:

$$\omega_m = \frac{\varepsilon_p V_{\phi p}}{\varepsilon_m V_{\phi m}}\omega_p - \frac{(K_{vp} + K_{vm})}{\varepsilon_m V_{\phi m}}\Delta p - \frac{V_0}{B_a \varepsilon_m V_{\phi m}}\frac{d\Delta p}{dt} \qquad (5.37)$$

5.2.4 Constant torque loading

We will analyse the characteristics of a transmission which is controlled by variation of pump flow delivery and which is subjected to constant torque $T_m = T_{load} = const.$ When we vary ε_p and maintain $\varepsilon_m = const.$, the pressure in the system is:

$$T_m = T_{tm}\eta_{hmm} = \varepsilon_m V_{\phi m}\Delta p_{34}\eta_{hmm} \qquad (5.38)$$

$$\Delta p_{34} = \frac{T_m}{\eta_{hmm} V_{\phi m}\varepsilon_m} \qquad (5.39)$$

If we assume that hydro-mechanical efficiency η_{hmm} of the motor is independent of motor speed ω_m, then pressure Δp is constant:

$$\frac{d\Delta p}{dt} = 0 \qquad (5.40)$$

Thus the last term in the equation for ω_m, eq. (5.37), is equal zero and substituting eq. (5.40) into eq. ((5.37)) we obtain:

$$\omega_m = \frac{\varepsilon_p V_{\phi p}}{\varepsilon_m V_{\phi m}}\omega_p - \frac{(K_{vp} + K_{vm})T_m}{\varepsilon_m^2 V_{\phi m}^2 \eta_{hmm}} \qquad (5.41)$$

Looking at this equation we can see that as torque T_m increases the speed of the motor decreases due to volumetric losses in the pump and the motor. Fig. 139 shows the effect of volumetric losses on the speed of the motor when displacement parameter ε_p is varied, **a** represents system volumetric loss when $\omega_p = 0$ and **b** is a decrease of motor speed due to volumetric losses.

5.2.5 Inertial Loading

The following two cases are considered:

Case **a**. Inertial loading - compressibility of fluid and viscous friction are ignored

Case **b**. Inertial loading - compressibility of fluid is taken into account

Case **c**. Inertial loading - compressibility of fluid and viscous friction are included

The input in each case is a step change in pump displacement parameter ε_p. Pump angular speed ω_p is assumed to be constant. We assume that the load on the motor is in the form:

$$T_{load} = T_m = J\frac{d\omega_m}{dt} \qquad (5.42)$$

and using the definition for hydro-mechanical efficiency of the motor we obtain:

$$\Delta p_{34} = \frac{T_m}{\eta_{hmm}\varepsilon_m V_{\phi m}} = \frac{J}{\eta_{hmm}\varepsilon_m V_{\phi m}}\frac{d\omega_m}{dt} \tag{5.43}$$

when we introduce this relation into the equation for ω_m, eq. (5.37), the following equation for motor speed is obtained:

$$\omega_m = \frac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}}\varepsilon_p - \frac{J(K_{vp}+K_{vm})}{\varepsilon_m^2 V_{\phi m}^2 \eta_{hmm}}\frac{d\omega_m}{dt} - \frac{V_0 J}{B_a \varepsilon_m^2 V_{\phi m}^2 \eta_{hmm}}\frac{d^2\omega_m}{dt^2} \tag{5.44}$$

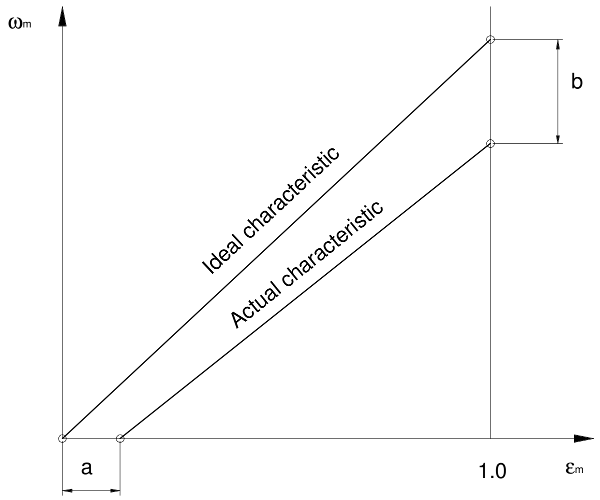

Fig. 139. Effect of volumetric losses on the speed of the motor when the displacement parameter ε_p is varied

Case a. As in this analysis compressibility of the fluid is ignored ($B_a = \infty$), the above equation takes the following form:

$$\frac{J(K_{vp}+K_{vm})}{\varepsilon_m^2 V_{\phi m}^2 \eta_{hmm}}\frac{d\omega_m}{dt} + \omega_m = \frac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}}\varepsilon_p \tag{5.45}$$

We can define time constant τ:

$$\tau = \frac{J(K_{vp}+K_{vm})}{\varepsilon_m^2 V_{\phi m}^2 \eta_{hmm}} \tag{5.46}$$

We can observe that if a system has large inertial load J and large volumetric losses characterised by coefficients K_{vp} and K_{vm} this will result in large time constant τ and thus in slow response of the system.

Constant k:

$$k = \frac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}} \tag{5.47}$$

and we can now rewrite eq. (5.45) as:

$$\tau \frac{d\omega_m}{dt} + \omega_m = k\varepsilon_p \tag{5.48}$$

which is a first order differential equation. With step change in pump displacement parameter e_p and with initial values of $\omega_m(0) = 0$ and $\frac{d\omega_m}{dt} = 0$ the above equation after transformation into Laplace domain becomes:

$$\omega_m(s) = \frac{ke_p}{s\,(\tau s + 1)} = \frac{ke_p/\tau}{s\,(s + 1/\tau)} \tag{5.49}$$

which after expansion into partial fractions and transformation back into time domain yields expression for time response of the motor to step change in e.g. ε_p):

$$\omega_m(t) = \omega_o \left[1 - e^{-t/\tau}\right] \tag{5.50}$$

where ω_o is a steady-state speed of the motor:

$$\omega_o = \frac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}} \varepsilon_p \tag{5.51}$$

Case b. We will use again eq. (5.44) and this time compressibility of the fluid is included in the analysis. Eq. (5.44) after some rearranging becomes:

$$\frac{d^2\omega_m}{dt^2} + \frac{B_a(K_{vp} + K_{vm})}{V_0}\frac{d\omega_m}{dt} + \frac{B_a\varepsilon_m^2 V_{\phi p}^2 \eta_{hmm}}{JV_0}\omega_m = \frac{B_a\varepsilon_m^2 V_{\phi m}^2 \eta_{hmm}}{JV_0}\frac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}}\varepsilon_p \tag{5.52}$$

we notice that this equation is a second order differential equation and thus it can be written as follows:

$$\ddot{\omega}_m + 2\xi\omega_n \dot{\omega}_m + \omega_n^2 \omega_m = k\varepsilon_p \tag{5.53}$$

where:

ω_n	-	natural frequency of the system
ζ	-	damping ratio

Comparing eq. (5.52) with eq. (5.53) we find that natural frequency ω_n of the system is equal to:

$$\omega_n = \varepsilon_m V_{\phi m} \sqrt{\frac{\eta_{hmm} B_a}{J V_0}} \tag{5.54}$$

and the damping ratio ζ is expressed by:

$$\zeta = \frac{B_a (K_{vp} + K_{vm})}{2V_0} \frac{1}{\omega_n} = \frac{B_a (K_{vp} + K_{vm})}{2V_0 \varepsilon_m V_{\phi m}} \sqrt{\frac{J V_0}{\eta_{hmm} B_a}} \tag{5.55}$$

which after rearranging becomes:

$$\zeta = \frac{J\,(K_{vp} + K_{vm})}{\varepsilon_m^2 V_{\phi m}^2 \eta_{hmm}} \frac{\varepsilon_m V_{\phi m}}{2} \sqrt{\frac{\eta_{hmm} B_a}{J V_0}} \tag{5.56}$$

and finally, constant k is equal to:

$$k = \frac{B_a \varepsilon_m^2 V_{\phi m}^2 \eta_{hmm}}{J V_0} \frac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}}$$

The equation for damping ratio ζ shows that its value will increase as volumetric losses K_{vp} and K_{vm} in transmission units and load moment of inertia J increase. Thus amplitudes of damped oscillations of the motor speed will be reduced during the transient state. The frequency of damped oscillations is expressed by:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{5.57}$$

The solution of equation (5.52) is in the form:

$$\omega_m = \omega_0 \left[1 - e^{-\zeta \omega_n t} \left\{ \cos(\omega_d t) + \frac{\zeta \omega_n}{\omega_d} \sin(\omega_d t) \right\} \right] \tag{5.58}$$

and it represents time response of speed ω_m of the motor. Steady-state speed of the motor ω_0 is equal to:

$$\omega_0 = \frac{k \varepsilon_p}{\omega_n^2} = \frac{\dfrac{\eta_{hmm} B_a \varepsilon_m^2 V_{\phi m}^2}{J V_0} \dfrac{V_{\phi p}}{\varepsilon_m V_{\phi m}}}{\left(\varepsilon_m V_{\phi m} \sqrt{\dfrac{\eta_{hmm} B_a}{J V_0}} \right)^2} = \frac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}} \tag{5.59}$$

Case c. Solving a more general loading case, i.e. assuming that the load on the motor includes viscous friction, then load equation is:

$$T_{load} = T_m = J \frac{d\omega_m}{dt} + f \omega_m \tag{5.60}$$

where f is viscous friction coefficient. Pressure differential across the motor is then represented by:

$$\Delta p_{34} = \frac{T_m}{\eta_{hmm}\varepsilon_m V_{\phi m}} = \frac{J}{\varepsilon_m V_{\phi m}\eta_{hmm}}\frac{d\omega_m}{dt} + \frac{f}{\varepsilon_m V_{\phi m}\eta_{hmm}}\omega_m \tag{5.61}$$

and its derivative by:

$$\frac{d\Delta p_{34}}{dt} = \frac{J}{\varepsilon_m V_{\phi m}\eta_{hmm}}\frac{d^2\omega_m}{dt^2} + \frac{f}{\varepsilon_m V_{\phi m}\eta_{hmm}}\frac{d\omega_m}{dt} \tag{5.62}$$

We substitute the above equations into eq. (5.37) and obtain:

$$\begin{aligned}\omega_m \;=\; & \frac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}}\varepsilon_p - \frac{(K_{vp}+K_{vm})}{\varepsilon_m V_{\phi m}}\left(\frac{J}{\varepsilon_m V_{\phi m}\eta_{hmm}}\frac{d\omega_m}{dt} + \frac{f}{\varepsilon_m V_{\phi m}\eta_{hmm}}\omega_m\right) - \\ & - \frac{V_0}{B_a\varepsilon_m V_{\phi m}}\left(\frac{J}{\varepsilon_m V_{\phi m}\eta_{hmm}}\frac{d^2\omega_m(t)}{dt^2} + \frac{f}{\varepsilon_m V_{\phi m}\eta_{hmm}}\frac{d\omega_m(t)}{dt}\right)\end{aligned} \tag{5.63}$$

which after rearranging becomes:

$$k_1\frac{d^2\omega_m}{dt^2} + k_2\frac{d\omega_m}{dt} + k_3\omega_m = k_4 e_p \tag{5.64}$$

where constant k_1:

$$k_1 = \frac{V_0 J}{B_a\varepsilon_m^2 V_{\phi m}^2\eta_{hmm}} \tag{5.65}$$

constant k_2:

$$k_2 = \frac{(K_{vp}+K_{vm})\,J + \dfrac{V_0 f}{B_a}}{\eta_{hmm}\varepsilon_m^2 V_{\phi m}^2} \tag{5.66}$$

constant k_3:

$$k_3 = 1 + \frac{(K_{vp}+K_{vm})\,f}{\varepsilon_m^2 V_{\phi m}^2\eta_{hmm}} \tag{5.67}$$

and finally coefficient k_4 is equal to:

$$k_4 = \frac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}} \tag{5.68}$$

As eq. (5.64) represents again a second order differential equation, we can again write it as:

$$\ddot{\omega}_m(t) + 2\xi\omega_n \dot{\omega}_m(t) + \omega_n^2\omega_m(t) = ke_p(t) \tag{5.69}$$

then natural frequency ω_n is equal to:

$$\begin{aligned} \omega_n &= \sqrt{\frac{k_3}{k_1}} = \left[\frac{\left(1.0 + \dfrac{(K_{vp}+K_{vm})f}{\eta_{hmm}\varepsilon_m^2 V_{\phi m}^2}\right)}{\dfrac{V_0}{B_a\varepsilon_m^2 V_{\phi m}^2}\dfrac{J}{\eta_{hmm}}}\right]^{1/2} = \\ &= \sqrt{\frac{\left(\varepsilon_m^2 V_{\phi m}^2\eta_{hmm} + fK_{vp} + fK_{vm}\right)B_a}{V_0 J}} \end{aligned} \tag{5.70}$$

damping ζ is expressed by equation:

$$\begin{aligned} \zeta &= \frac{k_2\sqrt{k_1}}{2k_1\sqrt{k_3}} = .5\frac{k_2}{\sqrt{k_1k_3}} = \frac{0.5\left(\dfrac{(K_{vp}+K_{vm})\,J + \dfrac{V_0 f}{B_a}}{\varepsilon_m^2 V_{\phi m}^2\eta_{hmm}}\right)}{\sqrt{\left(\dfrac{V_0 J}{B_a\varepsilon_m^2 V_{\phi m}^2\eta_{hmm}}\right)\left(1.0 + \dfrac{(K_{vp}+K_{vm})\,f}{\varepsilon_m^2 V_{\phi m}^2\eta_{hmm}}\right)}} \\ &= \frac{1}{2}\frac{JB_aK_{vp} + JB_aK_{vm} + V_0 f}{\sqrt{\left(\varepsilon_m^2 V_{\phi m}^2\eta_{hmm} + fK_{vp} + fK_{vm}\right)B_aV_0J}} \end{aligned} \tag{5.71}$$

and k is expressed by:

$$k = \frac{k_4}{k_1} = \frac{\dfrac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}}}{\dfrac{V_0 J}{B_a\varepsilon_m^2 V_{\phi m}^2\eta_{hmm}}} = \frac{\omega_p V_{\phi p}\varepsilon_m V_{\phi m} B_a \eta_{hmm}}{V_0 J} \tag{5.72}$$

Finally, time response of motor speed ω_m to step input e_p is described by equation:

$$\omega_m = \omega_0\left[1 - e^{-\zeta\omega_n t}\left\{\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d}\sin(\omega_d t)\right\}\right] \tag{5.73}$$

where steady-state speed of the motor ω_0 is equal to:

$$\omega_0 = \frac{k\varepsilon_p}{\omega_n^2} = \frac{\frac{\omega_p V_{\phi p}\varepsilon_m V_{\phi m} B_a \eta_{hmm}}{V_0 J}}{\frac{\left(\varepsilon_m^2 V_{\phi m}^2 \eta_{hmm} + f K_{vp} + f K_{vm}\right) B_a}{V_0 J}} = \frac{\frac{\omega_p V_{\phi p}}{\varepsilon_m V_{\phi m}}}{1 + \frac{\left(K_{vp} + K_{vm}\right) f}{\varepsilon_m^2 V_{\phi m}^2 \eta_{hmm}}}\varepsilon_p \tag{5.74}$$

and damped frequency as before is calculated using:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{5.75}$$

The values of ω_d, ω_n and ζ are defined by modified expressions which reflect the effect of friction coefficient f. Time responses of motor speed in a hydraulic transmission subjected to various loading conditions are shown in fig. 140.

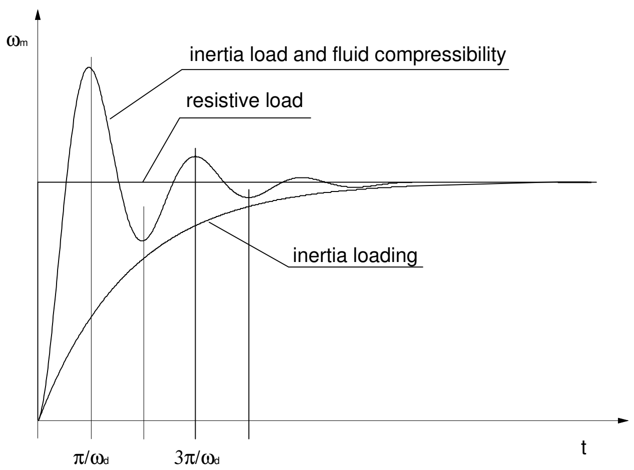

Fig. 140. Time responses of motor speed in a hydraulic transmission

Further, a more realistic analysis of hydrostatic transmissions and their dynamics, taking into consideration non-linearities in the system, requires application of digital simulation techniques.

Examples of Calculations - Hydrostatic transmissions

Problem 5.1 Hydrostatic transmission - open circuit

The following data is available for a hydraulic system shown in fig. 141:

- pump volumetric efficiency $\eta_{vp} = 0.95$
- pump hydro-mechanical efficiency $\eta_{hmp} = 0.97$
- line volumetric efficiency $\eta_{vl} = 1$
- line hydraulic efficiency $\eta_{hl} = 0.90$
- motor volumetric efficiency $\eta_{vm} = 0.95$
- motor hydro-mechanical efficiency $\eta_{hmm} = 0.96$
- pump rotational speed $n_p = 1475$ rpm
- pump input torque $T_p = 100$ Nm

Assume that pump and motor displacements are the same, i.e. $V_{\phi m} = V_{\phi p}$, and using the above data calculate:

- volumetric efficiency η_v and hydro-mechanical efficiency η_{hm} of the system
- system overall efficiency η
- rotation speed n_m and torque T_m of the motor
- kinematic i_k and dynamic i_d transmission ratios of the system

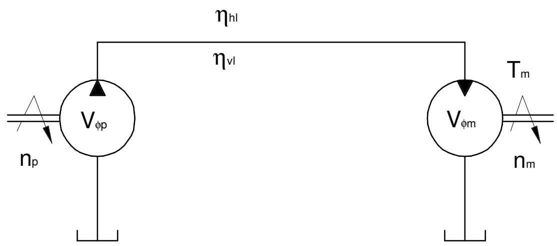

Fig. 141. Open circuit hydrostatic drive (Problem 5.1)

Answer: We find the volumetric efficiency of the system from equation:

$$\eta_v = \eta_{vp}\eta_{vl}\eta_{vm} = 0.95 \times 1.0 \times 0.95 = 0.902 \qquad \text{answer!}$$

and the hydro-mechanical efficiency:

$$\eta_{hm} = \eta_{hmp}\eta_{hl}\eta_{hmm} = 0.97 \times 0.9 \times 0.96 = 0.838 \qquad \text{answer!}$$

thus, the overall system efficiency is:

$$\eta = \eta_v \eta_{hm} = 0.902 \times 0.838 = 0.756 \qquad \text{answer!}$$

As we assumed that the volumetric efficiency of the hydraulic line is $\eta_{vl} = 1$, then $Q_m = Q_p$ and the rotation speed of the motor is:

$$\begin{aligned} n_m &= \eta_{vm}\eta_{vp}n_p\frac{V_{\phi p}}{V_{\phi m}} \\ n_m &= 0.95 \times 0.95 \times 1475 \times 1.0 = 1331\ rpm \qquad \text{answer!} \end{aligned}$$

We use the following equation to calculate motor torque:

$$\eta_{hmp} = \frac{T_{tp}}{T_p} = \frac{\Delta p_{12} V_{\phi p}}{T_p}$$

so pressure differential across the pump is equal to:

$$\Delta p_{12} = \frac{\eta_{hmp} T_p}{V_{\phi p}}$$

on the other hand:

$$\eta_{hmm} = \frac{T_m}{T_{tm}} = \frac{T_m}{V_{\phi m} \Delta p_{34}}$$

and also:

$$\eta_{hl} = \frac{\Delta p_{34}}{\Delta p_{12}}$$

So, the motor torque is equal to:

$$\begin{aligned} T_m &= \eta_{hmm}\Delta p_{34} V_{\phi m} = \eta_{hmm}\eta_{hl}\eta_{hmp}T_p\frac{V_{\phi m}}{V_{\phi p}} \\ T_m &= 0.96 \times 0.9 \times 0.97 \times 1.00 = 83.8 \text{ Nm} \qquad \text{answer!} \end{aligned}$$

The kinematic transmission ratio is defined by the relation:

$$\begin{aligned} i_k &= \frac{n_m}{n_p} = \frac{\eta_{vm}\eta_{vp}n_p\frac{V_{\phi p}}{V_{\phi m}}}{n_p} = \eta_{vm}\eta_{vp}\frac{V_{\phi p}}{V_{\phi m}} \\ i_k &= 0.95 \times 0.95 \times 1 = 0.90 \qquad \text{answer!} \end{aligned}$$

whereas the dynamic transmission ratio is equal to:

$$i_d = \frac{T_m}{T_p}$$

therefore:

$$i_d = \frac{83.8}{100} = 0.84 \qquad \text{answer!}$$

Problem 5.2 Valve control of hydrostatic transmission - open circuit

The following data is available for a hydraulic system shown in fig. 142:

- pump volumetric efficiency $\eta_{vp} = 0.95$
- pump hydro-mechanical efficiency $\eta_{hmp} = 0.964$
- motor volumetric efficiency $\eta_{vm} = 0.95$
- motor hydro-mechanical efficiency $\eta_{hmm} = 0.93$
- Pressure losses in the system are:
  - pressure loss Δp_1 in the line connecting the pump to the motor, $\Delta p_1 = 0.3$ MPa
  - pressure loss Δp_2 in the line connecting the motor to the reservoir, $\Delta p_2 = 0.1$ MPa
  - pressure loss Δp_3 in the return line filter, $\Delta p_3 = 0.05$ MPa
  - pressure loss Δp_4 in the suction line filter, $\Delta p_2 = 0.01$ MPa
- rotation speed of the electric motor driving pump is constant at $n_p = 2900$ rpm
- load torque on the motor is 280 Nm at rotation speed $n_m = 600$ rpm
- the relief valve is set at $p_0 = 15$ MPa.

Calculate unit displacements of the pump and the motor as well as the power of the electric motor.

Answer: The unit displacement $V_{\phi m}$ of a motor may be calculated from equation:

$$V_{\phi m} = \frac{T_m}{\eta_{hmm} \Delta p_{34}}$$

When we allow the motor to work with the maximum allowable pressure difference Δp_{34} we obtain the minimum value of the motor displacement, i.e. the smallest motor size. As the relief valve is set at 15 MPa the maximum value of Δp_{34} is obtained from equation:

$$\begin{aligned} \Delta p_{34} &= p_0 - \Delta p_1 - \Delta p_2 - \Delta p_3 \\ \Delta p_{34} &= 15 - 0.3 - 0.1 - 0.05 = 14.55 \text{ MPa} \end{aligned}$$

thus:

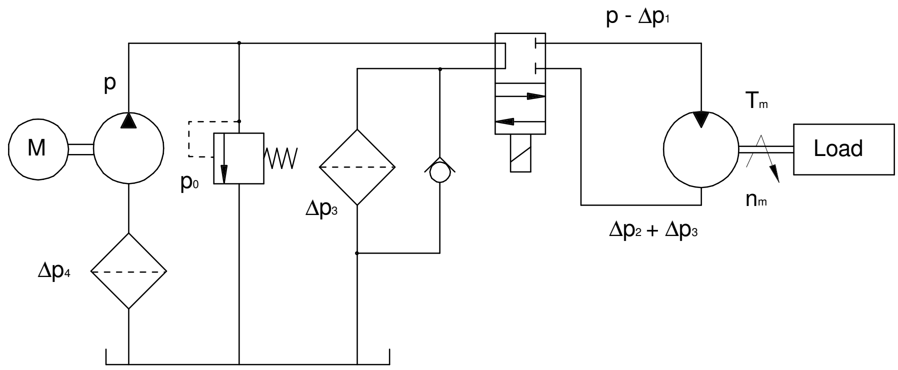

Fig. 142. Valve controlled hydrostatic drive (Problem 5.2)

$$V_{\phi m} = \frac{T_m}{\eta_{hmm}\Delta p_{34}} = \frac{280}{0.93 \times 14.55 \times 10^6} = 20.7 \times 10^{-6}\ \mathrm{m^3 rad^{-1}} \qquad \text{answer!}$$

If we assume that the relief valve is closed then the motor demand flow is equal to the pump delivery flow:

$$V_{\phi p}\frac{2\pi n_p}{60}\eta_{vp} = \frac{V_{\phi m}}{\eta_{vm}}\frac{2\pi n_m}{60}$$

and the unit displacement of the pump is then equal to:

$$V_{\phi p} = \frac{n_m}{n_p}\frac{V_{\phi m}}{\eta_{vm}\eta_{vp}} = \frac{600}{2900} \times \frac{20.7 \times 10^{-6}}{0.95 \times 0.95} = 4.74 \times 10^{-6}\ \mathrm{m^3 rad^{-1}} \qquad \text{answer!}$$

The power of the electric motor is equal to:

$$P_{el} = T_p\omega_p = \frac{V_{\phi p}\Delta p_{12}}{\eta_{hmp}}\frac{2\pi n_p}{60}$$

where pressure difference Δp_{12} is equal to:

$$\begin{aligned} \Delta p_{12} &= p_0 + \Delta p_4 \\ \Delta p_{12} &= 15 + 0.01 = 15.01\ \mathrm{MPa} \end{aligned}$$

thus the power of the electric motor is:

$$P_{el} = \frac{4.74 \times 10^{-6} \times 15.01 \times 10^6 \times 2900}{0.946} \times \frac{2\pi}{60} \times \frac{1}{1000} = 22.8\,\mathrm{kW} \qquad \text{answer!}$$

Power of the electric motor can be also calculated from equation:

$$P_{el} = \frac{P_{load}}{\eta} = \frac{T_{load} n_m}{\eta_{vp}\eta_{hmp}\eta_{hl}\eta_{vm}\eta_{hmm}} \frac{2\pi}{60}$$

where hydraulic line efficiency η_{hl} is equal to:

$$\eta_{hl} = \frac{\Delta p_{34}}{\Delta p_{12}} = \frac{14.55}{15.01} = 0.969$$

thus again the electric power required to drive the pump is:

$$P_{el} = \frac{280 \times 600}{0.95 \times 0.946 \times 0.969 \times 0.95 \times 0.93} \times \frac{2\pi}{60} \frac{1}{1000} = 22.8\,\text{kW} \qquad \text{answer!}$$

Problem 5.3 Hydrostatic transmission for mining lift

A simplified diagram of a hydraulic system driving a mining lift is shown in fig. 143. The following data is available:

- maximum tension in the lifting rope is $S_{max} = 40000$ N
- radius of the drum is $r = 0.4\,\text{m}$
- maximum lift velocity is $v_{max} = 2\ \text{ms}^{-1}$
- rotation speed of the pump is $n_p = 30\ \text{revs}^{-1}$
- maximum permissible pressure differential across the pump is $\Delta p_{12} = 23.2\,\text{MPa}$.

The following data on efficiencies is available:

- pump volumetric efficiency $\eta_{vp} = 0.96$
- pump hydro-mechanical efficiency $\eta_{hmp} = 0.93$
- line volumetric efficiency $\eta_{vl} = 0.98$
- line hydra-mechanical efficiency $\eta_{hl} = 0.95$
- motor volumetric efficiency $\eta_{vm} = 0.94$
- motor hydro-mechanical efficiency $\eta_{hmm} = 0.91$

Using above data calculate:

- unit displacement of the motor $V_{\phi p}$
- power of electric motor P_{el}

Answer: The unit displacement of the motor is defined by equation:

$$V_{\phi m} = \frac{T_m}{\Delta p_{34}\eta_{hmm}}$$

and the motor load torque is:

$$T_m = T_{load} = S_{max} \times r = 4 \times 10^4 \times 0.4 = 1.6 \times 10^4\ \text{Nm}$$

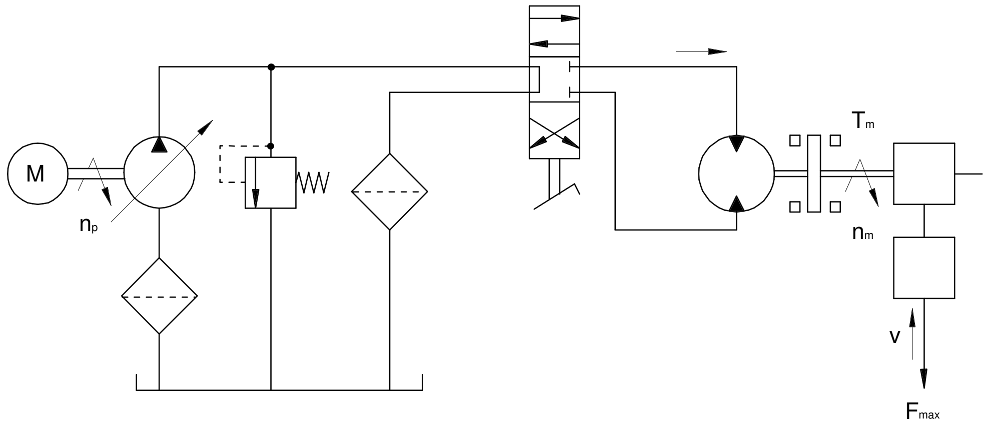

Fig. 143. Hydraulic mining lift (Problem 5.3)

The pressure difference across the hydraulic motor is calculated taking hydro-mechanical efficiency of the hydraulic line into account:

$$\Delta p_{34} = \Delta p_{12}\eta_{hl} = 23.2 \times 0.95 = 22 \text{ MPa}$$

thus motor displacement is:

$$V_{\phi m} = \frac{1.6 \times 10^4}{22 \times 10^6 \times 0.91} = 7.99 \times 10^{-4} \text{ m}^3\text{rad}^{-1}$$

The maximum unit displacement of the pump is calculated using the relation between the pump and motor flows and taking into consideration the volumetric efficiency of the hydraulic line:

$$Q_p = \frac{Q_m}{\eta_{vl}}$$

and as the equation for motor flow demand is:

$$Q_m = \frac{\omega_m V_{\phi m}}{\eta_{vm}}$$

and the pump flow delivery is equal to:

$$Q_p = \omega_p V_{\phi p} \eta_{vp}$$

then the equation for unit displacement of the pump is:

$$V_{\phi p} = \frac{V_{\phi m}}{\eta_{vm}\eta_{vl}\eta_{vp}} \frac{\omega_m}{\omega_p} \tag{a}$$

The motor rotation speed is equal to:

$$\omega_m = \frac{v_{max}}{r} = \frac{2}{0.4} = 5\ \text{rads}^{-1}$$

and the pump rotation speed is equal to:

$$\omega_p = 2\pi n_p = 2\pi \times 30 = 188.5\ \text{rads}^{-1}$$

Thus using eq. (a) we obtain unit displacement of the pump:

$$V_{\phi p} = \frac{7.99 \times 10^{-4}}{0.94 \times 0.98 \times 0.96} \times \frac{5}{188.5} = 2.4 \times 10^{-5}\ \text{m}^3\text{rad}^{-1} \qquad \text{answer!}$$

We calculate power of the electric motor using equation:

$$\begin{aligned} P_{el} &= \frac{T_{load}\omega_m}{\eta} = \frac{T_{load}\omega_m}{\eta_{hmp}\eta_{vp}\eta_{hl}\eta_{vl}\eta_{hmm}\eta_{vm}} = \\ &= \frac{1.6 \times 10^4 \times 5}{0.93 \times 0.96 \times 0.95 \times 0.98 \times 0.91 \times 0.94 \times 10^3} = 112.5\,\text{kW} \quad \text{answer!} \end{aligned}$$

Problem 5.4 Hydraulic transmission for snowmobile

A simplified diagram of a hydraulic transmission for a snowmobile is shown in fig. 144. Each crawler is driven by a separate hydraulic motor. The following data is available:

- mass of the vehicle is $M = 4 \times 10^3$ kg
- radius of drive wheels is $r = 0.2$ m
- the prime mover is a diesel engine which delivers maximum power at $n = 3600$ rpm
- the maximum pressure difference across the pump is $\Delta p_{max} = 15$ MPa
- maximum speed of vehicle is $v_1 = 36$ kph on flat terrain and $v_2 = 7.2$ kph on 30% gradient
- total driving resistance F_T is assumed constant over the range of driving conditions, $F_T = 4 \times 10^3$ N

Calculate, ignoring losses:

- unit displacement of hydraulic motor $V_{\phi m}$
- maximum unit displacement of hydraulic pump $V_{\phi p}$
- power of diesel engine P_{en}

Carry out the same calculations for the case when the hydraulic system efficiencies are:

- pump volumetric efficiency $\eta_{vp} = 0.96$
- pump hydro-mechanical efficiency $\eta_{hmp} = 0.92$
- motor volumetric efficiency $\eta_{hmp} = 0.95$

- motor hydro-mechanical efficiency $\eta_{hmm} = 0.90$

Values of pressure losses in hydraulic lines and in the directional control valve should be taken from the circuit diagram.

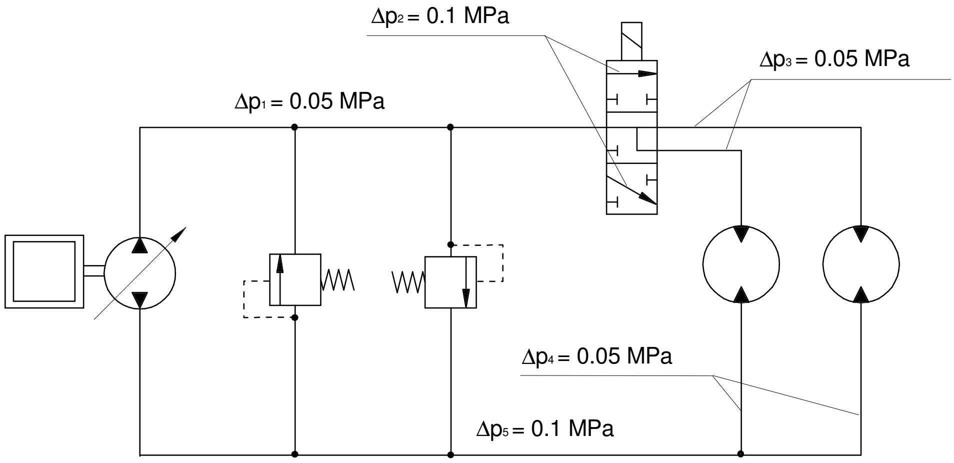

Fig. 144. Hydraulic drive of a snowmobile (Problem 5.4)

Answer: The maximum unit displacement of the motor is calculated when the maximum load torque and the pressure difference across the motor are known using equation:

$$\Delta p_{34} = \frac{T_m}{V_{\phi m}}$$

The maximum load torque occurs when the vehicle travels the 30% gradient, thus maximum total force opposing motion of the snowmobile is equal to:

$$F = Mg \sin\alpha + F_T \tag{a}$$

and $\sin\alpha \approx \tan\alpha = 0.3$ (gradient 30%). As drive motors work in parallel then the torque required to drive the snowmobile is:

$$2T_m = Fr$$

and using eq. (a) we obtain:

$$\begin{aligned} T_m &= \frac{r}{2}(Mg \sin\alpha + F_T) = \\ &= \frac{0.2}{2} \times (4000 \times 9.81 \times 0.3 + 4000) = 1577 \text{ Nm} \end{aligned}$$

thus the required motor unit displacement is equal to:

$$V_{\phi m} = \frac{T_m}{\Delta p_{34}} = \frac{1577}{150 \times 10^5} = 1.05 \times 10^{-4}\ \mathrm{m^3 rad^{-1}} \qquad \text{answer!}$$

When the maximum pump delivery at maximum speed of the vehicle is known, the maximum unit displacement of the pump can be calculated using:

$$V_{\phi p} = \frac{Q_p}{\omega_p} = \frac{60 Q_p}{2\pi n_p}$$

and as maximum rotation speed of the motors is equal to:

$$\omega_m = \frac{v_1}{r} = \frac{36 \times 10^3}{3600 \times 0.2} = 50\ \mathrm{rads^{-1}}$$

then the required flow delivery capacity of the pump is:

$$Q_p = 2Q_m = 2\omega_m V_{\phi m} = 2 \times 50 \times 1.05 \times 10^{-4} = 0.0105\ \mathrm{m^3 s^{-1}}$$

thus the required unit displacement of the pump at maximum input speed of $n =$ 3600 rpm is:

$$V_{\phi p} = \frac{60 \times 0.0105}{2\pi \times 3600} = 27.85 \times 10^{-5}\ \mathrm{m^3 rad^{-1}} \qquad \text{answer!}$$

The power required to propel the vehicle at maximum speed on flat ground is:

$$P_1 = F_T v_1 = 4 \times 10^3 \times \frac{36 \times 10^3}{3600 \times 1000} = 40\,\mathrm{kW}$$

and the power required when travelling the 30% gradient is equal to:

$$\begin{aligned} P_2 &= (Mg \sin\alpha + F_T) v_2 = \\ &= (4 \times 10^3 \times 9.81 \times 0.3 + 4 \times 10^3) \times \frac{7.2 \times 10^3}{3600} = 31.54\,\mathrm{kW} \end{aligned}$$

From the above calculations we see that, when losses in the system are not taken into consideration, the power developed by the diesel engine should be:

$$P_{en} \geqslant P_1 = 40.0\,\mathrm{kW} \qquad \text{answer!}$$

When we take the system pressure losses into consideration then the maximum pressure differential on the hydraulic motor is equal to:

$$\Delta p_{34} = 15 - (0.05 + 0.1 + 0.05 + 0.05 + 0.1) = 14.65\ \mathrm{MPa}$$

and, as motor hydro-mechanical efficiency is $\eta_{hmm} = 0.90$, then the required unit displacement of each motor is equal to:

$$V_{\phi m} = \frac{T_m}{\Delta p_{34}\eta_{hmm}} = \frac{1577}{14.65 \times 10^6 \times 0.9} = 120 \times 10^{-6}\ \mathrm{m^3rad^{-1}} \qquad \text{answer!}$$

The pump unit displacement is calculated from equation:

$$V_{\phi p} = \frac{Q_p}{\omega_p \eta_{vp}}$$

where:

$$Q_p = 2Q_m = 2\frac{\omega_m V_{\phi m}}{\eta_{vm}}$$

thus:

$$V_{\phi p} = \frac{2\omega_m V_{\phi m}}{\omega_p \eta_{vp}\eta_{vm}} = \frac{2 \times 50 \times 120 \times 10^{-6}}{3600 \times \frac{2\pi}{60} \times 0.96 \times 0.95} = 34.9 \times 10^{-6}\ \mathrm{m^3rad^{-1}} \qquad \text{anwer!}$$

Overall efficiency of the transmission is expressed by:

$$\eta = \eta_{vp}\eta_{hmp}\eta_{vm}\eta_{hmm}\frac{\Delta p_{34}}{\Delta p_{max}} \qquad \text{(b)}$$

thus, taking losses into consideration, the required engine power should be at least equal to:

$$P_{en} = \frac{P_1}{\eta} = \frac{4.0 \times 10^4}{0.96 \times 0.92 \times 0.95 \times 0.90 \times \frac{14.65}{15.0} \times 1000} = 54.23\,\mathrm{kW} \qquad \text{answer!}$$

Problem 5.5 Closed circuit hydrostatic transmission

A closed circuit hydrostatic transmission shown in fig. 145 is assembled using the following major components:

- a variable displacement hydraulic pump with stroke displacement $q_p = 0.08$ Lrev^{-1}
- an electric motor which has rotation speed $n_p = 1000$ rpm
- a variable displacement hydraulic motor subjected to constant load $P_{load} = 12$ kW
- a relief valve set at $p_0 = 27$ MPa

Efficiencies and pressure losses in the system are as follows:

- pump volumetric efficiency is $\eta_{vp} = 0.90$
- electric motor efficiency is $\eta_{el} = 0.97$

- system overall efficiency is $\eta = 0.60$
- motor volumetric efficiency is $\eta_{vm} = 0.93$
- motor hydro-mechanical efficiency is $\eta_{hmm} = 0.97$
- pressure loss in the line connecting the pump with the motor is $\Delta_{loss} = 0.8$ MPa.

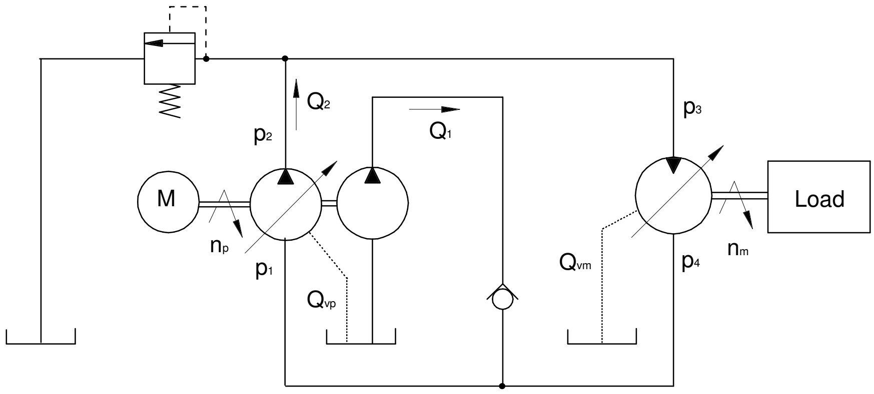

Fig. 145. Closed circuit hydrostatic drive (Problem 5.5)

Assuming that the above efficiencies are constant over the full operating range and that pump displacement parameter is ε= 0.5, calculate:

- delivery flow Q_1 of the charge pump, necessary to avoid system cavitation.
- the percentage increase in system power prior to the opening of the relief valve.
- the maximum stroke displacement of the hydraulic motor if its minimum rotation speed is $n_m = 200$ rpm
- power of the electric motor P_{el}
- hydro-mechanical efficiency of the pump η_{hmm}

Answer: The actual flow delivery of the main pump is:

$$Q_2 = \varepsilon_p n_p q_p \eta_{vp}$$
$$Q_2 = 0.5 \times 1000 \times 80 \times 10^{-3} \times 0.9 \times \frac{1}{60} = 0.6 \text{ Ls}^{-1}$$

If we assume that motor return pressure $p_3 = 0$ and that, initially, the relief valve is closed i.e. motor flow demand $Q_m = Q_2$, then the motor delivery pressure p_{34} is:

$$p_3 = \frac{P_{load}}{Q_m \eta_{hmm} \eta_{vm}} = \frac{12}{0.6 \times 0.93 \times 0.97} = 24.7 \text{ MPa}$$

and the required pump pressure p_2 is equal to:

$$p_2 = p_3 + \Delta p_{loss} = 24.7 + 0.8 = 25.5 \text{ MPa} < p_0 = 27 \text{ MPa}$$

The calculated value of pressure p_2 is lower than the relief valve setting, thus the assumption that the relief valve is closed was correct. We assume that volumetric efficiency of the pump is constant over its full operating range, then pump volumetric losses are:

$$Q_{vp} = (1 - \eta_{vp})Q_{tp}$$

where Q_{tp} is theoretical pump flow:

$$Q_{tp} = \varepsilon_p n_p q_p$$

thus:

$$Q_{vp} = (1 - \eta_{vp})\varepsilon_p n_p q_p = \frac{(1 - 0.9) \times 0.5 \times 1000 \times 80 \times 10^{-3}}{60} = 66.7 \times 10^{-3}\ \mathrm{Ls}^{-1}$$

In a similar way we may calculate the volumetric losses in the hydraulic motor:

$$Q_{vm} = (1 - \eta_{vm})Q_m = (1 - 0.93) \times 0.6 = 42.0 \times 10^{-3}\ \mathrm{Ls}^{-1}$$

We may further assume that volumetric losses due to internal leakages in the pump and motor are drained to the reservoir, thus a charge pump must provide a flow at least equal to:

$$Q_1 = Q_{vp} + Q_{vm} = 66.7 \times 10^{-3} + 42.0 \times 10^{-3} = 108.7 \times 10^{-3}\ \mathrm{Ls}^{-1} \qquad \text{answer!}$$

The maximum pump pressure, set by the relief valve, is $p_0 = 27$ MPa and if we consider pressure loss in the delivery line then the maximum pressure at the hydraulic motor is:

$$\begin{aligned} p_{3max} &= p_0 - \Delta p_{loss} \\ p_{3max} &= 27 - 0.8 = 26.2\,\mathrm{MPa} \end{aligned}$$

thus the maximum power which motor can deliver is:

$$P_{max} = \frac{p_{3max}}{p_3} P_{load} = \frac{26.2}{24.7} \times 12 = 12.73\,\mathrm{kW}$$

and the percentage power overload causing opening of the relief valve is:

$$\begin{aligned} P &= \frac{P_{max} - P_{load}}{P_{load}} \times 100 \\ P\% &= \frac{12.73 - 12}{12} \times 100 = 6.1\% \qquad \text{answer!} \end{aligned}$$

The maximum stroke displacement of the hydraulic motor, at minimum motor speed $n_m = 200$ rpm and $\varepsilon_m = 1$, is equal to:

$$q_m = \frac{Q_2 \eta_{vm}}{n_m} = \frac{0.6 \times 0.93 \times 60}{200} = 167 \times 10^{-3}\ \mathrm{Lrev}^{-1} \qquad \text{answer!}$$

Input pump power is equal to:

$$P_p = \frac{P_{load}}{\eta} = \frac{12}{0.6} = 20.0\,\text{kW}$$

thus the required power of the electric motor is:

$$P_{el} = \frac{P_p}{\eta_{el}} = \frac{20.0}{0.89} = 22.5\,\text{kW} \qquad \text{answer!}$$

Finally, the hydro-mechanical efficiency of the pump is defined by equation:

$$\eta_{hmp} = \frac{p_2 \varepsilon_p V_{\phi p}}{T_p} = \frac{p_2 \varepsilon_p n_p q_p}{P_p}$$

$$\eta_{hmp} = \frac{25.5 \times 10^6 \times 0.5 \times \dfrac{1000}{60} \times 0.08}{20 \times 10^3 \times 10^3} = 0.85 \qquad \text{answer!}$$

Problem 5.6 Acceleration of hydrostatic transmission

A hydrostatic transmission consists of a variable displacement pump and a fixed displacement hydraulic motor. Following data is available:

- stroke displacement of the pump is $q_p = 164 \times 10^{-3}$ Lrev^{-1}
- the pump is driven by an electric motor at a constant rotation speed $n_p =$ 25 revs^{-1}
- pump leakage coefficient is $K_{vp} = 9 \times 10^{-3}$ Ls^{-1}MPa^{-1}
- pump hydro-mechanical efficiency is $\eta_{hmp} = 0.85$
- motor inertial load is $J = 1.0\,\text{kg m}^2$
- hydraulic motor has stroke displacement $q_m = 65 \times 10^{-3}$ Lrev^{-1}
- motor leakage coefficient is $K_{vm} = 8 \times 10^{-3}$ Ls^{-1}MPa^{-1}and its hydro-mechanical efficiency is $\eta_{hmm} = 0.85$.

Ignore volumetric and hydro-mechanical losses in the hydraulic lines and the compressibility of the fluid and calculate the following:

- the acceleration of the hydraulic motor, when its rotation speed $n_m = 33$ revs^{-1}and pump flow reaches 60% of full flow
- required output power of the electric motor.
- the time taken by the hydraulic motor to reach 63% of its top speed when pump displacement parameter has a step change from $\varepsilon_p = 0$ to $\varepsilon_p = 0.5$.

Answer: With the assumption, that volumetric losses in the hydraulic lines and compressibility of the fluid can be ignored, the following equation is valid:

$$Q = \varepsilon_p n_p q_p - K_{vp} \Delta p_{12} = n_m q_m + K_{vm} \Delta p_{34}$$

and if we ignore pressure losses then:

$$\Delta p_{12} = \Delta p_{34} = \Delta p$$

thus when pump displacement setting is $\varepsilon_p = 0.6$ (60% full theoretical flow) and motor speed is 33 revs^{-1}the pressure differential across the motor is equal to:

$$\Delta p = \frac{\varepsilon_p n_p q_p - n_m q_m}{K_{vp} + K_{vm}} \qquad \text{(a)}$$

$$\Delta p = \frac{0.6 \times 25 \times 164 \times 10^{-3} - 33 \times 65 \times 10^{-3}}{(9 \times 10^{-3} + 8 \times 10^{-3})} = 18.5 \text{ MPa}$$

Torque of the motor due to inertial load is equal:

$$T_{load} = T_m = J \frac{d\omega_m}{dt}$$

and also:

$$T_m = \frac{\eta_{hmm} q_m \Delta p}{2\pi}$$

if we compare these equations we obtain the equation for acceleration of the motor:

$$J \frac{d\omega_m}{dt} = \frac{\eta_{hmm} q_m \Delta p}{2\pi}$$

$$\frac{d\omega_m}{dt} = \frac{\eta_{hmm} q_m \Delta p}{2\pi J} \qquad \text{(b)}$$

and

$$\frac{d\omega_m(t)}{dt} = \frac{0.85 \times 17.5 \times 10^6 \times 65 \times 10^{-6}}{2\pi \times 1} = 154 \text{ rads}^{-2} \qquad \text{answer!}$$

The required output power of the electric motor is:

$$P_{el} = P_p = \frac{\Delta p Q_p}{\eta_p} = \frac{\Delta p \varepsilon_p n_p q_p \eta_{vp}}{\eta_{hmp} \eta_{vp}} = \frac{\Delta p \varepsilon_p n_p q_p}{\eta_{hmp}}$$

$$P_{el} = \frac{18.5 \times 10^6 \times 0.6 \times 25 \times 164 \times 10^{-6}}{0.85 \times 1000} = 53.5\,\text{kW} \qquad \text{answer!}$$

Considering equations (a) and (b); the equation for motor motion is:

$$\frac{d\omega_m}{dt} = \frac{\eta_{hmm} q_m}{2\pi J} \frac{\varepsilon_p n_p q_p - n_m q_m}{K_{vp} + K_{vm}}$$

rearranging we obtain:

$$\frac{J(K_{vp}+K_{vm})2\pi}{\eta_{hmm}q_m}\frac{d\omega_m(t)}{dt}=\varepsilon_p n_p q_p - n_m q_m$$

which after following substitutions:

$$\begin{aligned} q_p &= 2\pi V_{\phi p} \\ q_m &= 2\pi V_{\phi m} \\ n_p &= \frac{\omega_p}{2\pi} \\ n_m &= \frac{\omega_m}{2\pi} \end{aligned}$$

and division by of $V_{\phi m}$ becomes:

$$\frac{J(K_{vp}+K_{vm})}{\eta_{hmm}V_{\phi m}^2}\frac{d\omega_m(t)}{dt}+\omega_m(t)=\varepsilon_p\frac{V_{\phi p}}{V_{\phi m}}\omega_p$$

which is a first order differential equation. Using Laplace transformations, with initial conditions equal to zero, we obtain the following equation:

$$\tau s\omega_m(s)+\omega_m(s)=\varepsilon_p\omega_p\frac{V_{\phi p}}{V_{\phi p}}$$

where time constant τ, is equal to:

$$\tau=\frac{J(K_{vp}+K_{vm})}{\eta_{hmm}V_{\phi m}^2}=\frac{1\times(9\times10^{-6}+8\times10^{-6})\times10^{-6}\times(2\pi)^2}{0.85\times(65\times10^{-6})^2}=0.187\,\mathrm{s}$$

Thus the equation for motor speed in Laplace domain is:

$$\omega_m(s)=\varepsilon_p\omega_p\frac{V_{\phi p}}{V_{\phi m}}\frac{1}{(\tau s+1)}$$

which, after transformation back into the time domain, yields the equation for motor speed:

$$\omega_m(t)=\varepsilon_p\omega_p\frac{V_{\phi p}}{V_{\phi m}}\left(1-e^{-\frac{t}{\tau}}\right)$$

The maximum, steady-state, motor speed is obtained setting $t=\infty$

$$\omega_{m(max)}=\omega_m(\infty)=\varepsilon_p\omega_p\frac{V_{\phi p}}{V_{\phi m}}$$

thus, for step change of ε_p from $0\longrightarrow0.5$, steady-state speed of the motor is:

$$\omega_{m(max)} = \varepsilon_p \omega_p \frac{V_{\phi p}}{V_{\phi m}} = 0.5 \times 25 \times \frac{164 \times 10^{-6}}{65 \times 10^{-6}} = 31.5\, revs^{-1}$$

The time in which the motor reaches 63% of its maximum speed is obtained by equating the expressions:

$$\begin{aligned} \omega_m(t) &= 0.63\omega_{m(max)} \\ \varepsilon_p \omega_p \frac{V_{\phi p}}{V_{\phi m}}(1 - e^{-\frac{t}{\tau}}) &= 0.63\varepsilon_p \omega_p \frac{V_{\phi p}}{V_{\phi m}} \\ (1 - e^{-\frac{t}{\tau}}) &= 0.63 \longrightarrow e^{-\frac{t}{\tau}} = 0.37 \end{aligned}$$

therefore $t \approx \tau$ and finally, the time in which the motor will reach 63% of its maximum speed is:

$$t = \tau = \frac{J(K_{vp}+K_{vm})}{\eta_{hmp} V_{\phi m}^2} = 0.187\,\mathrm{s} \qquad \text{answer!}$$

Problem 5.7 Natural frequency of hydrostatic transmission

A hydrostatic transmission consists of a variable displacement pump and a fixed displacement hydraulic motor. Following data is available:

- unit displacement of the pump is $V_{\phi p} = 18 \times 10^{-3}$ Lrad^{-1}
- pump is driven by an electric motor at constant rotation speed $\omega p = 12$ rads^{-1}
- pump leakage coefficient $K_{vp} = 4 \times 10^{-3}$ Ls^{-1}MPa^{-1}
- pump hydro-mechanical efficiency $\eta_{hmp} = 1.0$
- motor inertial load $J = 2$ kgm^2
- hydraulic motor has a unit displacement $V_{\phi m} = 12 \times 10^{-3}$ Lrad^{-1}, leakage coefficient for pump and motor are $K_{vp} = K_{vm} = 4 \times 10^{-3}$ Ls^{-1}MPa^{-1}and hydro-mechanical efficiency $\eta_{hmm} = 0.90$
- compressibility of the fluid $B = 1.7 \times 10^3$ MPa.
- total fluid volume in the system $V_0 = 2$ L, volumetric and hydraulic losses in the hydraulic lines are ignored.

Using above data calculate:

- system natural frequency f
- damping coefficient ζ
- magnitude of moment of inertia J at which damping coefficient will be $\zeta = 1$.

Answer: Motion of the motor in this system is represented by eq. (5.52) which takes into consideration volumetric losses in the pump and the motor and the compressibility of fluid. This equation, with $\varepsilon_m = 1$ becomes:

$$\frac{d^2\omega_m}{dt^2} + \frac{B_a(K_{vp} + K_{vm})}{V_0}\frac{d\omega_m}{dt} + \frac{B_a\varepsilon_m^2 V_{\phi p}^2 \eta_{hmm}}{JV_0}\omega_m = \frac{B_a\varepsilon_m^2 V_{\phi m}^2 \eta_{hmm}}{JV_0}\frac{\omega_p V_{\phi p}}{V_{\phi m}}\varepsilon_p \quad \text{(a)}$$

The system's natural frequency is:

$$\omega_n = V_{\phi m}\sqrt{\frac{\eta_{hmm}B}{JV_0}} \quad \text{(b)}$$

$$\omega_n = 12 \times 10^{-6}\sqrt{\frac{0.9 \times 1.7 \times 10^9}{2 \times 2 \times 10^{-3}}} = 7.42\ \text{rads}^{-1} \quad \text{(answer!)}$$

and damping coefficient ζ is equal to:

$$\zeta = \frac{B(K_{vp} + K_{vm})}{2V_0\omega_n} \quad \text{(c)}$$

$$\zeta = \frac{1.7 \times 10^3 \times (4 + 4) \times 10^{-3}}{2 \times 2 \times 7.42} = 0.458$$

Damped natural frequency of the system is equal to:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = 7.42\sqrt{1 - 0.458^2} = 6.6\ \text{rads}^{-1} \quad \text{(answer!)}$$

or:

$$f = \frac{\omega_d}{2\pi} = 1.05\,\text{Hz}$$

We assume that $\zeta = 1$ thus using eq. (c) and eq. (b) we obtain:

$$\zeta = \frac{B(K_{vp} + K_{vm})}{2V_0\omega_n} = \frac{B(K_{vp} + K_{vm})}{2V_0V_{\phi m}}\sqrt{\frac{JV_0}{\eta_{hmm}B}} = 1$$

thus the required value of moment of inertia J is:

$$J = \frac{4V_0V_{\phi m}^2\eta_{hmm}}{B(K_{vp} + K_{vm})^2}$$

$$J = \frac{4 \times 2 \times 10^{-3} \times \left(12 \times 10^{-6}\right)^2 \times 0.9}{1.7 \times 10^9 \times (4 \times 10^{-6} + 4 \times 10^{-6})^2} = 9.5\ \text{kgm}^2 \quad \text{answer!}$$

Problem 5.8 Damping in hydrostatic transmission

A hydraulic system in which the speed response of the hydraulic motor to a step change in the pump displacement parameter ε_p (from 0 to 1) is described by eq.

(5.58). Calculate the ratios between successive speed amplitudes $\frac{\omega_0}{\omega_1}, \frac{\omega_1}{\omega_2}, ..., \frac{\omega_k}{\omega_{k+1}}$ as functions of damping coefficient ζ . Assume angular speed of the pump ω_p and unit displacement $V_{\phi m}$ of the motor to be constant. Steady state speed of the motor is ω_0.

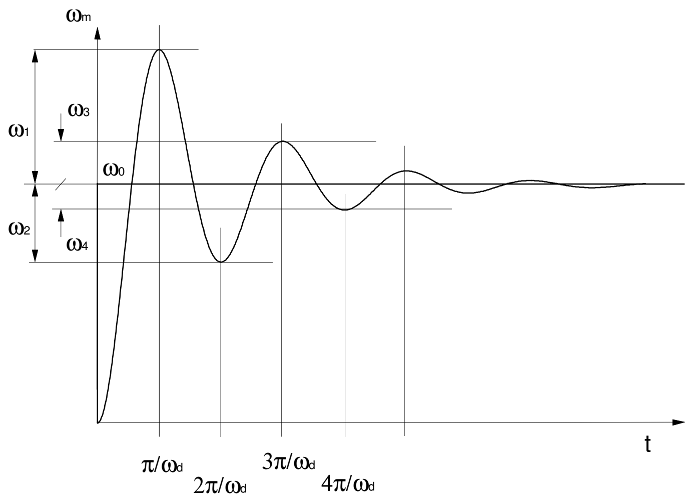

Fig. 146. Relations between successive oscillations of motor speed (Problem 5.8)

Answer: We use equations (a) and (c) from the previous problem in (b) and write the following equation for motor speed ω_m:

$$\frac{d^2\omega_m}{dt^2} + 2\zeta\omega_n\frac{d\omega_m}{dt} + \omega_n^2\omega_m(t) = \omega_n^2\omega_p\frac{V_{\phi p}}{V_{\phi m}}\varepsilon_p$$

where:

ω_n - natural frequency of the system
ζ - damping coefficient.

We assume a unit step change in the pump displacement ε_p, thus:

$$\varepsilon_p = \begin{array}{ll} 1 & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t \leq 0 \end{array}$$

For $t = 0$, $\omega_m(0) = 0$ and $\frac{d\omega_m}{dt} = 0$. Transforming into the Laplace domain we obtain:

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)\omega_m(s) = \omega_n^2\omega_p\frac{V_{\phi p}}{V_{\phi m}}\frac{1}{s}$$

we write:

$$\frac{\omega_p V_{\phi p}}{V_{\phi m}} = \omega_0$$

and after rearranging we obtain the equation for motor speed (in s-domain):

$$\omega_m(s) = \frac{\omega_0 \omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

After performing partial fraction expansion we obtain:

$$\omega_m(s) = \omega_0 \left(\frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n + \omega_n^2} \right)$$

or in a different form:

$$\omega_m(s) = \omega_0 \left(\frac{1}{s} - \frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \right)$$

where damped natural frequency is equal to:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

After factorising we obtain:

$$\omega_m(s) = \omega_0 \left(\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \right)$$

which after transforming back into the time domain using following transforms:

$$\begin{aligned}
&\frac{1}{s} \overset{T}{\Longrightarrow} 1 \\
&\frac{1}{(s-a)^2 + b^2} \overset{T}{\Longrightarrow} \frac{1}{b} e^{at} \sin(bt) \\
&\frac{s-a}{(s-a)^2 + b^2} \overset{T}{\Longrightarrow} e^{at} \cos(bt)
\end{aligned}$$

we finally obtain the equation for motor speed in the time domain:

$$\omega_m(t) = \omega_0 \left\{ 1 - e^{-\zeta\omega_n t} \left[\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_d} \sin(\omega_d t) \right] \right\}$$

This equation describes the change of motor speed in response to step change in pump displacement parameter ε_p from 0 to 1, fig. 146. The successive amplitudes of motor speed ω_k for $k = 1, 2 \ldots n$ can be obtained from the above equation by inserting $t = \dfrac{k\pi}{\omega_d}$

$$\omega_k = \left|\omega_m\left(k\frac{\pi}{\omega_d}\right) - \omega_0\right|$$

and as motor speed for each k is:

$$\omega_m\left(k\frac{\pi}{\omega_d}\right) = \omega_0\left\{1 - e^{-\zeta\omega_n\frac{k\pi}{\omega_d}}\left[\cos\left(\omega_d\frac{k\pi}{\omega_d}\right) + \frac{\zeta\omega_n}{\omega_d}\sin\left(\omega_d\frac{k\pi}{\omega_d}\right)\right]\right\}$$

and noting that:

$$\begin{aligned}\cos\left(\omega_d\frac{k\pi}{\omega_d}\right) &= \begin{cases}-1 & \text{for } k \text{ odd integer}\\ +1 & \text{for } k \text{ even integer}\end{cases}\\ \sin\left(\omega_d\frac{k\pi}{\omega_d}\right) &= 0\end{aligned}$$

we obtain:

$$\omega_k = \left|\omega_0\left[1 - e^{-k\pi\zeta\frac{\omega_n}{\omega_d}}\right] - \omega_0\right| = e^{-k\pi\zeta\frac{\omega_n}{\omega_d}}$$

an finally, the ratio of amplitudes is:

$$\left|\frac{\omega_k}{\omega_{k+1}}\right| = e^{\pi\zeta\frac{\omega_n}{\omega_d}} = e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \qquad \text{answer!}$$

Problem 5.9 Hydraulic stiffness of cylinder drive system

Calculate hydraulic stiffness C of a system shown in fig. 147 and determine the position of the piston for which stiffness C is minimum. Assume that bulk modulus B of the fluid is constant and that the cylinder walls and hydraulic lines are perfectly rigid.

Answer: The stiffness of a hydraulic system is defined by equation:

$$C = \frac{\Delta F}{\Delta x} \tag{a}$$

where:
$\Delta F, \Delta x$ - change in force magnitude and position

The change in force magnitude is equal to:

$$\Delta F = A_1\Delta p_1 + A_2\Delta p_2 \tag{b}$$

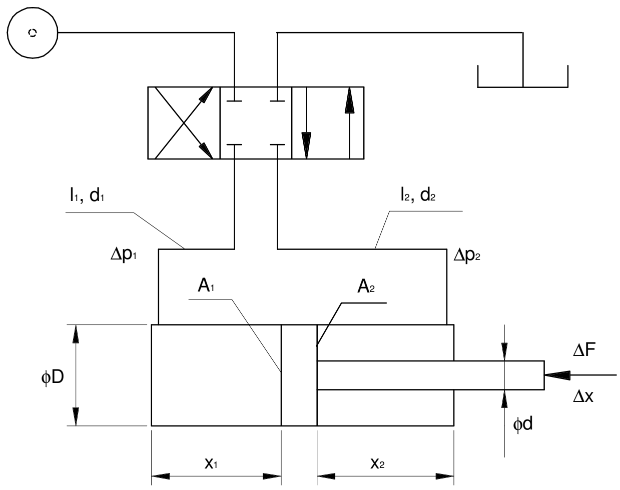

Fig. 147. Hydraulic system (Problem 5.9)

where:
A_1, A_2 - piston areas

As we assumed that the compliance of the cylinder and hydraulic lines are to be ignored, changes of pressure p_1 in the piston head volume and pressure p_2 in the annulus volume are defined by the equations:

$$\Delta p_1 = \frac{B\Delta V_1}{V_1 + V_{l1}} \qquad \Delta p_2 = \frac{B\ \Delta V_2}{V_2\ + V_{l2}} \tag{c}$$

where:
B -
V_1, V_2 - piston head and annulus volumes
V_{l1}, V_{l2} - volumes of hydraulic lines
l_1, l_2 - length of hydraulic lines
d_1, d_2 - internal diameters of hydraulic lines

Volumes of the cylinder and hydraulic lines are:

$$V_1 = A_1 x_1 \qquad V_{l1} = \frac{\pi d_1^2}{4} l_1$$
$$V_2 = A_2 x_2 \qquad V_{l2} = \frac{\pi d_2^2}{4} l_2$$

As areas A_1 and A_2 are constant, then:

$$\Delta V_1 = A_1 \Delta x \qquad \Delta V_2 = A_2 \Delta x$$

and using eq. (a) , (b) and (c) we obtain:

$$C = \frac{\Delta F}{\Delta x} = B\left(\frac{A_1^2}{V_1 + V_{l1}} + \frac{A_2^2}{V_2 + V_{l2}}\right)$$

Let us assume that:

$$\begin{aligned} A_1 &= 2A_2, \\ d_1 &= d_2 \\ l_1 &= l_2 \end{aligned}$$

then volumes of hydraulic lines are:

$$V_{l1} = V_{l2} = V_l$$

and we also denote $\frac{V_l}{A_1} = b$ and $x_1 + x_2 = h$. Then after substitution in eq. we obtain:

$$\begin{aligned} C &= B\left(\frac{A_1^2}{A_1 x_1 + A_1 b} + \frac{A_1^2}{4\left(\frac{A_1(h-x_1)}{2} + A_1 b\right)}\right) = \\ C &= \frac{BA_1}{2}\frac{2h - x_1 + 5b}{(x_1+b)(h-x_1+2b)} \end{aligned} \qquad \text{(d)}$$

Minimum stiffness of the system occurs when:

$$\begin{aligned} \frac{dC}{dx_1} &= 0 \\ \frac{dC}{dx_1} &= \frac{d}{dx_1}\left(\frac{BA_1}{2}\frac{2h - x_1 + 5b}{x_1(h+b) - x_1^2 + b(h+2b)}\right) = 0 \\ \frac{dC}{dx_1} &= -\frac{1}{2}BA_1\frac{-4x_1 h - 10x_1 b + x_1^2 + 8bh + 7b^2 + 2h^2}{\left(x_1 h + x_1 b - x_1^2 + bh + 2b^2\right)^2} = 0 \end{aligned}$$

and after rearranging we obtain equation:

$$x_1^2 - x_1(4h + 10b) + 2h^2 + 8hb + 7b^2 = 0$$

this is a quadratic equation, a discriminant Δ is equal to:

$$\Delta = (4h + 10b)^2 - 4(2h^2 + 8hb + 7b^2) = 8h^2 + 48bh + 72b^2$$

square root of Δ is equal to:

$$\sqrt{\Delta} = \sqrt{8h^2 + 48bh + 72b^2} = 2\sqrt{2}\sqrt{\left((3b + h)^2\right)}$$

or in simpler form:

$$\sqrt{\Delta} = \sqrt{8}h + \sqrt{72}b$$

thus:

$$\begin{aligned} x_1 &= (2 - \sqrt{2})h + (5 - 3\sqrt{2})b \\ \frac{x_1}{h} &= 2 - \sqrt{2} + (5 - 3\sqrt{2})\frac{b}{h} = 0.585 - 0.757\frac{b}{h} \end{aligned} \tag{e}$$

if fluid volume in hydraulic lines is much smaller than fluid volume in the cylinder:

$$\frac{V_l}{A_1 h} \ll 1$$

then:

$$\frac{b}{h} \ll 1 \longrightarrow b \ll h$$

and using eq. (e) we obtain:

$$\frac{x_l}{h} \approx 0.585$$

thus minimum stiffness of the system will occur when $x_1 \approx 0.585h$. Volume displaced by the piston over full stroke $h = x_1 + x_{2.}$ will be denoted as V, then:

$$h = \frac{V}{A_1}$$

finally, system stiffness C_{min} has its minimum value:

$$\begin{aligned} C_{\min} &= \frac{BA_1}{2}\frac{2h - x_1 + 5b}{(x_1 + b)(h - x_1 + 2b)} = \frac{BA_1}{2}\frac{(2h - 0.585h)}{0.585h(h - 0.585h)} \\ &= \frac{BA_1}{2}\frac{(2 - 0.585)\,A_1}{0.585(1 - 0.585)V} = 2.9\frac{BA_1^2}{V} \end{aligned} \qquad \text{answer!}$$